Birational geometry of Calabi-Yau pairs

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Birational Geometry Seminar
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Birational geometry of Calabi-Yau pairs

Joint with Alessio Corti and Alex Massarenti

(We always work over $\mathbb{C}$)

Motivation: Automorphisms of Smooth Hypersurfaces

$X = X_d \subset \mathbb{P}^{n+1}$ smooth hypersurface of degree d

Theorem (Matsumura-Monsky 1964)

If $(n, d) \neq (1, 3), (2, 4)$, then

$$\mathrm{Aut}(\mathbb{P}^{n+1}, X) \twoheadrightarrow \mathrm{Aut}(X).$$

- $C = X_3 \subset \mathbb{P}^2$ genus 1 curve ($\mathrm{Aut}(C) \cong C \rtimes \mathbb{Z}/d\mathbb{Z}$)
- $S = X_4 \subset \mathbb{P}^3$ K3 surface ($\mathrm{Aut}(S)$ discrete and possibly infinite)

In both cases, the image of $\mathrm{Aut}(\mathbb{P}^{n+1}, X) \to \mathrm{Aut}(X)$ is finite.

$C = X_3 \subset \mathbb{P}^2$ genus 1 curve

Theorem

- Every automorphism of C is induced by a Cremona transformation of the ambient $\mathbb{P}^2$.

$$1 \to \mathrm{Ine}(\mathbb{P}^2, C) \to \mathrm{Dec}(\mathbb{P}^2, C) \to \mathrm{Aut}(C) \to 1$$

- (Pan 2007) Generators for decomposition group $\mathrm{Dec}(\mathbb{P}^2, C)$
- (Blanc 2008) Generators for inertia group $\mathrm{Ine}(\mathbb{P}^2, C)$

$S = X_4 \subset \mathbb{P}^3$ K3 surface

Question (Gizatullin)

Is every automorphism of S induced by a Cremona transformation of the ambient space $\mathbb{P}^3$?

Examples (Oguiso 2012)

- $\mathrm{Aut}(S) \cong \mathbb{Z}$, and no nontrivial automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.
- $\mathrm{Aut}(S) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and every automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.

Example (Paiva-Quedo 2022)

$\mathrm{Aut}(S) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and no nontrivial automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.

$S = X_4 \subset \mathbb{P}^3$ K3 surface

Problem

To describe the decomposition group of $S \subset \mathbb{P}^3$

$$\mathrm{Dec}(\mathbb{P}^3, S) \ = \ \left\{\varphi \in \mathrm{Bir}(\mathbb{P}^3) \mid \varphi_* S = S\right\}$$

and its image in $\mathrm{Aut}(S)$

$\left(\mathbb{P}^3, S\right)$ is a Calabi-Yau pair

Calabi-Yau pairs

Definition (Calabi-Yau pair (X, D))

- X terminal projective variety
- D is a hypersurface $\sim -K_X$
- (X, D) is log canonical

Example

$(\mathbb{P}^n, D)$ where $D \subset \mathbb{P}^n$ is a smooth hypersurface of degree $n+1$

Calabi-Yau pairs

Definition (Calabi-Yau pair (X, D))

- X terminal projective variety
- D is a hypersurface $\sim -K_X$
- (X, D) is log canonical

Remark

(X, D) Calabi-Yau pair $\rightsquigarrow$ $\exists$ ω_D (unique up to scaling)

$$\operatorname{div}(\omega_D) = -D$$

Calabi-Yau pairs

Definition (Calabi-Yau pair (X, D))

- X terminal projective variety
- D is a hypersurface $\sim -K_X$ $\quad (\ D = -\mathrm{div}(\omega_D) \ )$
- (X, D) is log canonical

Definition (volume preserving map $(X, D_X) \dashrightarrow (Y, D_Y)$)

$f : X \dashrightarrow Y$ birational map $\rightsquigarrow$ $f_* : \ \Omega^n_{\mathbb{C}(X)/\mathbb{C}} \rightarrow \Omega^n_{\mathbb{C}(Y)/\mathbb{C}}$

If $f_*\omega_{D_X} = \omega_{D_Y}$ (up to scaling) then we say that

$$f : \ (X, D_X) \dashrightarrow (Y, D_Y) \ \text{ is volume preserving}$$

Calabi-Yau pairs

Remark (Valuative interpretation)

$$
\begin{array}{ccc}
 & W & \\
\swarrow & & \searrow \\
X & \overset{f}{\dashrightarrow} & Y
\end{array}
$$

$$\forall E \subset W, \quad a(E, K_X + D_X) = a(E, K_Y + D_Y)$$

Example

If $D \subset \mathbb{P}^n$ is a smooth hypersurface of degree $n+1$, and $f : X \to \mathbb{P}^n$ is a volume preserving blowup along a smooth center Z, then

$$Z \subset D \quad \text{and} \quad \operatorname{codim}_{\mathbb{P}^n}(Z) = 2.$$

Problem

Given a Calabi-Yau pair (X, D), to determine

$$\mathrm{Bir}(X, D) := \left\{ \varphi \in \mathrm{Bir}(X) \,\middle|\, \varphi : (X, D) \dashrightarrow (X, D) \textit{ is volume preserving} \right\}$$

Example

$D = D_4 \subset \mathbb{P}^3$ smooth K3 surface

$$\mathrm{Dec}(\mathbb{P}^3, D) = \left\{ \varphi \in \mathrm{Bir}(\mathbb{P}^3) \mid \varphi_* D = D \right\} = \mathrm{Bir}\big(\mathbb{P}^3, D\big)$$

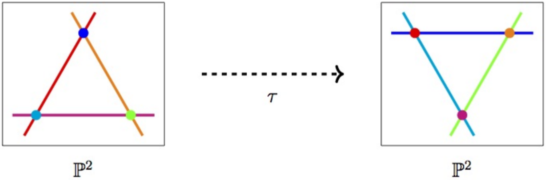
Remark

If (X, D) is a Calabi-Yau pair with canonical singularities, then

$$\mathrm{Dec}(X, D) \;=\; \left\{\varphi \in \mathrm{Bir}(X) \mid \varphi_* D = D\right\} \;=\; \mathrm{Bir}(X, D)$$

Example (Canonicity is necessary)

$$(X, D) \;=\; \left(\mathbb{P}^2, \sum_{i=0}^{2} H_i\right) \qquad \left(\omega_D = \frac{dx}{x} \wedge \frac{dy}{y} \right)$$



Remark

If (X, D) is a Calabi-Yau pair with canonical singularities, then

$$\mathrm{Dec}(X, D) = \left\{ \varphi \in \mathrm{Bir}(X) \mid \varphi_* D = D \right\} = \mathrm{Bir}(X, D)$$

Theorem (Blanc 2013)

$$\mathrm{Bir}\left(\mathbb{P}^2, \sum_{i=0}^{2} H_i\right) = \left\langle \underbrace{(\mathbb{C}^*)^2 \, , \, \mathrm{SL}(2, \mathbb{Z})}_{\text{preserve the torus } (\mathbb{C}^*)^2} \, , \, (x, y) \mapsto \left(y, \frac{1+y}{x}\right) \right\rangle$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x, y) \mapsto (x^a y^b, x^c y^d)$$

Problem

Given a Calabi-Yau pair (X, D), to determine $\mathrm{Bir}(X, D)$.

Theorem A

If (X, D) is terminal with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$ and $\mathrm{Pic}(D) = \mathbb{Z} \cdot \left(H_{|D}\right)$, then

$$\mathrm{Bir}(X, D) \; = \; \mathrm{Aut}(X, D).$$

Corollary

If $D \subset \mathbb{P}^n$ is a general hypersurface of degree $n + 1$ $(n \geq 3)$, then

$$\mathrm{Bir}\left(\mathbb{P}^n, D\right) \; = \; \mathrm{Aut}\left(\mathbb{P}^n, D\right).$$

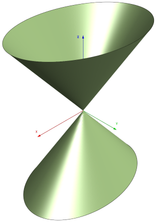
Theorem B

If $D \subset \mathbb{P}^3$ is a general quartic surface with one singular point, then

$$\mathrm{Bir}(\mathbb{P}^3, D) \cong \mathbb{G} \rtimes \mathbb{Z}/2\mathbb{Z}$$

$\mathbb{G}$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$

$$x_0^2 A_2(x_1, x_2, x_3) + x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0$$



$$\mathbb{G} = \Big\{ \big[(AG - BF)x_0 - CF : A(Fx_0 + G)x_1 : A(Fx_0 + G)x_2 : A(Fx_0 + G)x_3\big]$$

$$F, G \in \mathbb{C}[x_1, x_2, x_3] \text{ homogeneous with } \deg(G) = \deg(F) + 1 \Big\}$$

$D \subset \mathbb{P}^3$ general quartic hypersurface with one singular point P

$$x_0^2 A_2(x_1, x_2, x_3) + x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0$$

$$\begin{array}{ccccc} & & \tilde{D} \subset X & & \\ & \swarrow^{Bl_P} & & \searrow^{\mathbb{P}^1\text{-bundle}} & \\ D \subset \mathbb{P}^3 & & \overset{\pi_P}{\dashrightarrow} & & \mathbb{P}^2 \end{array} \qquad \begin{array}{ccc} \tilde{D} & \xrightarrow{\tau} & \tilde{D} \\ & \searrow \quad \swarrow & (2:1) \\ & \mathbb{P}^2 & \end{array}$$

$$\mathrm{Bir}(\mathbb{P}^3, D) \xrightarrow{r} \mathrm{Bir}(D) \cong \mathrm{Aut}(\tilde{D}) = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

Example

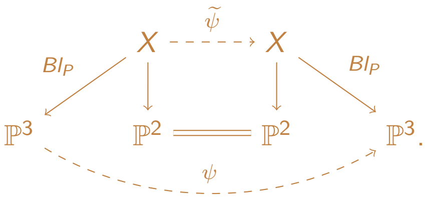
$$\varphi : (x_0 : x_1 : x_2 : x_3) \mapsto (-Ax_0 - B : Ax_1 : Ax_2 : Ax_3) \quad \rightsquigarrow \quad \tau$$

$$\boxed{1 \to \mathbb{G} \to \mathrm{Bir}(\mathbb{P}^3, D) \xrightarrow{\curvearrowleft} \mathbb{Z}/2\mathbb{Z} \to 1}$$

$D \subset \mathbb{P}^3$ general quartic hypersurface with 1 singular point P

$$1 \to \mathbb{G} \to \mathrm{Bir}(\mathbb{P}^3, D) \xrightarrow{\curvearrowleft} \mathbb{Z}/2\mathbb{Z} \to 1$$

Key point: Given $\psi \in \mathrm{Bir}(\mathbb{P}^3, D)$ there is a commutative diagram:



$\mathbb{G}$ is the group of birational self-maps of X over $\mathbb{P}^2$ fixing $\tilde{D}$ pointwise

View X as a model of $\mathbb{P}^1$ over $\mathbb{C}(x, y)$

$\mathbb{G}$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$ □

The Cremona Group

$$\mathrm{Bir}(\mathbb{P}^n) \;:=\; \left\{ \varphi : \mathbb{P}^n \; \overset{\sim}{\dashrightarrow} \; \mathbb{P}^n \;\; \text{birational self-map} \;\right\}$$

Example (The standard quadratic transformation)

$$\tau : \quad \begin{array}{ccc} \mathbb{P}^2 & \overset{\sim}{\dashrightarrow} & \mathbb{P}^2 \\ (x:y:z) & \longmapsto & \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right) = (yz : xz : xy) \end{array}$$

Theorem (Noether-Castelnuovo 1870-1901)

$$\mathrm{Bir}(\mathbb{P}^2) \;=\; \left\langle \; \mathrm{Aut}(\mathbb{P}^2) \;,\; \tau \; \right\rangle$$

Theorem (Hilda Hudson 1927)

For $n \geq 3$, $\mathrm{Bir}(\mathbb{P}^n)$ cannot be generated by elements of bounded degree.

The Sarkisov program (Corti 1995, Hacon-McKernan 2013)

$$\mathbb{P}^n = X_0 \overset{\psi_1}{\dashrightarrow} X_1 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{k-1}}{\dashrightarrow} X_{k-1} \overset{\psi_k}{\dashrightarrow} X_k = \mathbb{P}^n$$

$$\psi : X_0 \dashrightarrow X_k$$

The Sarkisov program (Corti 1995, Hacon-McKernan 2013)

$$\begin{array}{ccccccccc}
 & & & & \overset{\psi}{\dashrightarrow} & & & & \\
\mathbb{P}^n = X_0 & \overset{\psi_1}{\dashrightarrow} & X_1 & \overset{\psi_2}{\dashrightarrow} & \cdots & \overset{\psi_{k-1}}{\dashrightarrow} & X_{k-1} & \overset{\psi_k}{\dashrightarrow} & X_k = \mathbb{P}^n \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathrm{pt} & & Y_1 & & & & Y_{k-1} & & \mathrm{pt}
\end{array}$$

The $X_i \to Y_i$'s are Mori fiber spaces

- X_i has terminal singularities
- $\rho(X_i/Y_i) = 1$
- $-K_{X_i}$ is relatively ample

The ψ_i's are elementary links

The surface case

The Mori fiber spaces are:

- $\mathbb{P}^2 \to \text{pt}$
- $\mathbb{F}_m \to \mathbb{P}^1$ ($\mathbb{P}^1$-bundle)

($\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 \cong Bl_P\mathbb{P}^2$)

The elementary links are

Type 1

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow[Bl_P]{} & \mathbb{F}_1 \\ \downarrow & & \downarrow \\ \text{pt} & & \mathbb{P}^1 \end{array}$$

Type 2

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ \mathbb{F}_m & & & & \mathbb{F}_{m\pm1} \\ \downarrow & & & & \downarrow \\ \mathbb{P}^1 & & = & & \mathbb{P}^1 \end{array}$$

Type 3

$$\begin{array}{ccc} \mathbb{F}_1 & \xrightarrow{Bl_P} & \mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & & \text{pt} \end{array}$$

Type 4

$$\begin{array}{ccc} \mathbb{F}_0 & = & \mathbb{F}_0 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

Elementary links in higher dimensions

Type 1

Surfaces

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow{Bl_P} & \mathbb{F}_1 \\ \downarrow & & \downarrow \\ \text{pt} & \longleftarrow & \mathbb{P}^1 \end{array}$$

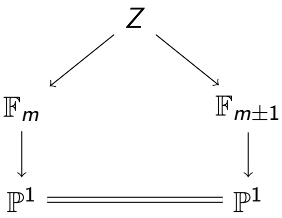
Higher dimensions

$$\begin{array}{ccccc} & & Z & \overset{\varphi}{\dashrightarrow} & X' \\ & \swarrow^{f} & & & \downarrow \\ X & & & & S' \\ \downarrow & & & \swarrow & \\ S & & & & \end{array}$$

Elementary links in higher dimensions

Type 2

Surfaces



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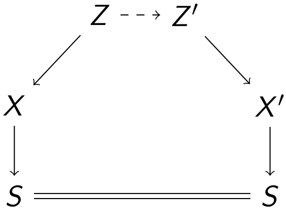

Volume Preserving Sarkisov Program

Theorem (Corti-Kaloghiros 2016)

A volume preserving birational map between Mori fibered Calabi-Yau pairs is a composition of volume preserving Sarkisov links .

$$
\begin{array}{ccccccccc}
 & & & & \overset{\psi}{\dashrightarrow} & & & & \\
(X_0, D_0) & \overset{\psi_1}{\dashrightarrow} & (X_1, D_1) & \overset{\psi_2}{\dashrightarrow} & \cdots & \overset{\psi_{k-1}}{\dashrightarrow} & (X_{k-1}, D_{k-1}) & \overset{\psi_k}{\dashrightarrow} & (X_k, D_k) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
Y_0 & & Y_1 & & & & Y_{k-1} & & Y_k
\end{array}
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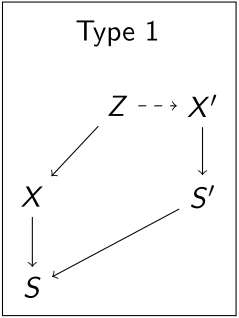

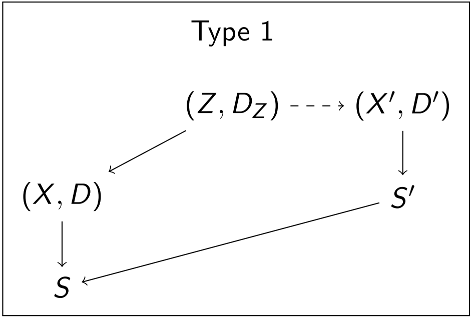

Theorem A

If $n \geq 3$ and D is a general hypersurface of degree $n+1$, then

$$\mathrm{Bir}\big(\mathbb{P}^n, D\big) \;=\; \mathrm{Aut}\big(\mathbb{P}^n, D\big).$$

$\big(\;\; D$ is smooth and $\mathrm{Pic}(D) = \mathbb{Z} \cdot \big(H_{|D}\big) \;\;\big)$

$$\begin{array}{ccccccccc}
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$$\overset{\psi}{\overbrace{\begin{array}{ccccccccc}
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\mathrm{pt} & & Y_1 & & & & Y_{k-1} & & \mathrm{pt}
\end{array}}}$$

X_1 has worst than terminal singularities

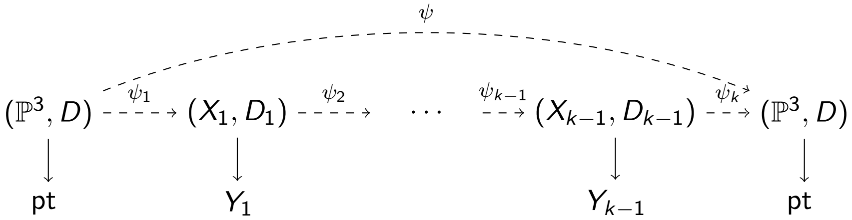
□

Theorem B

If $D \subset \mathbb{P}^3$ is a general quartic hypersurface with 1 singular point P, then

$$\mathrm{Bir}(\mathbb{P}^3, D) \;\cong\; \mathbb{G} \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $\mathbb{G}$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$.



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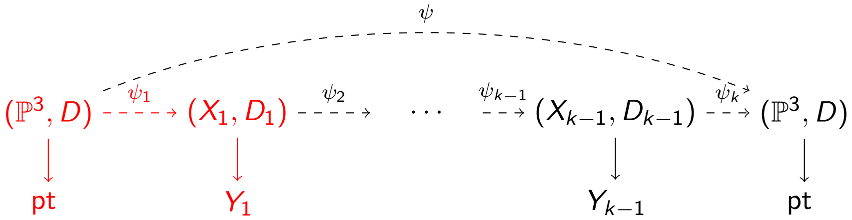

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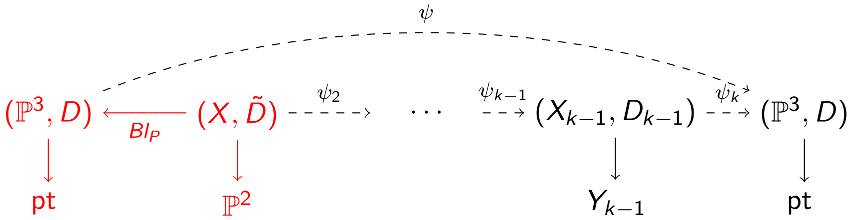

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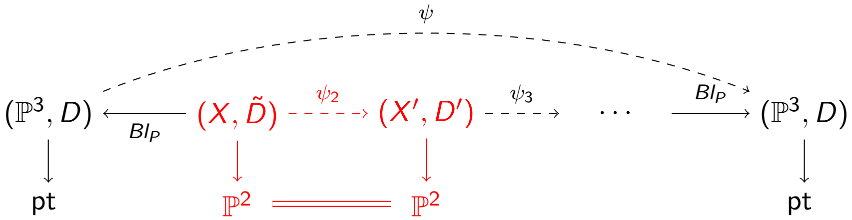

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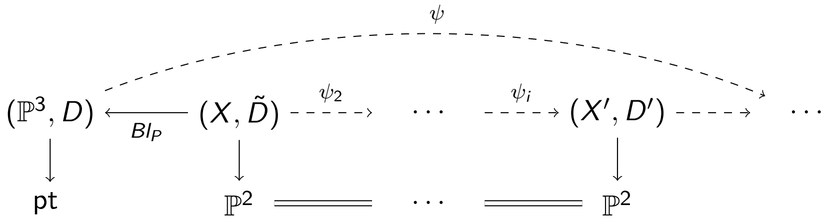

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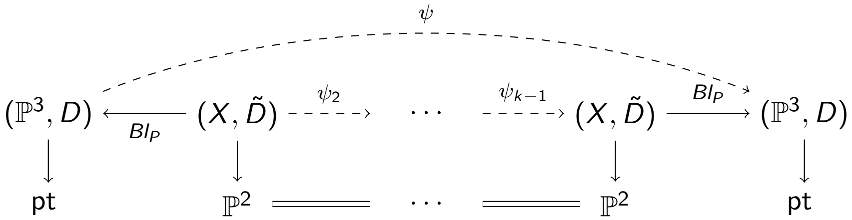

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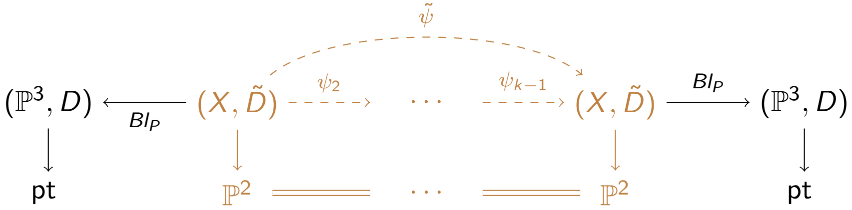


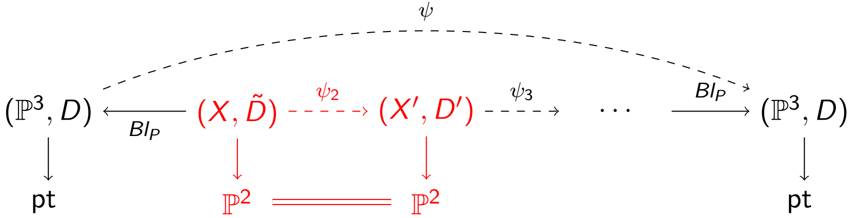
□

Definition (Pliability)

(X, D) Mori fibered Calabi-Yau pair

$$\mathcal{P}(X, D) \;:=\; \left\{ (X', D') \text{ Mf CY pair} \;\middle|\; \exists\, (X, D) \xrightarrow{\text{vol preserving}} (X', D') \right\} / \sim$$

Example (Square equivalence)



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Theorem C

If $D \subset \mathbb{P}^3$ general quartic hypersurface with one $A2$ singularity P, then we determine the pliability of $(\mathbb{P}^3, D)$:

- $(\mathbb{P}^3, D)$
- $\left(Bl_P \mathbb{P}^3, \tilde{D}\right) \to \mathbb{P}^2$
- $\left(\mathbb{P}(1^3, 2), D_5\right)$
- $\left(\mathbb{P}(1^3, 2), D_5'\right)$
- 3-parameter family $(X_4, D_{3,4})$, with $X_4 \subset \mathbb{P}(1^3, 2^2)$
- 6-parameter family $(X_4, D_{2,4})$, with $X_4 \subset \mathbb{P}(1^4, 2)$

Thank you!